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ALMOST SURE BEHAVIOR OF LINEAR FUNCTIONALS OF SUPERCRITICAL BRANCHING PROCESSES

BY

SØREN ASMUSSEN

ABSTRACT. The exact a.s. behavior of any linear functional $Z_n \cdot a$ of a supercritical positively regular p-type ($1 < p < \infty$) Galton-Watson process $\{Z_n\}$ is found under a second moment hypothesis. The main new results are of iterated logarithm type, with normalizing constants depending on the decomposition of a according to the Jordan canonical form of the offspring mean matrix.

1. **Introduction.** Consider a p-dimensional Galton-Watson process $\{Z_n\} = \{Z_n(1) \cdots Z_n(p)\}$. We introduce as briefly as possible the basic parameters and refer to Athreya and Ney [6, Chapter V] for additional background material. Let $\mathcal{I}_n$ be the set of individuals of the nth generation and, whenever $k \in \mathcal{I}_n$, let U_k be the offspring vector produced by k so that

$$Z_{n+1} = \sum_{k \in \mathcal{I}_n} U_k, \qquad n = 0, 1, 2, \ldots.$$

Specific assumptions on $\mathcal{I}_0$ are usually not relevant, but, whenever needed, we let P^i, E^i, Var^i, etc., refer to the case where $\mathcal{I}_0$ consists of one individual of type i. Letting $\mathfrak{F}_{n+1} = \sigma(U_k; k \in \mathcal{I}_m; m \leqslant n)$ we see that Z_{n+1} is $\mathfrak{F}_{n+1}$-measurable and the basic branching property states that, for fixed n, the U_k, $k \in \mathcal{I}_n$, are independent conditioned upon $\mathfrak{F}_n$ with

$$P(U_k \in A | \mathfrak{F}_n) = P^i(Z_1 \in A), \qquad A \subseteq \mathbf{N}^p,$$

$i = i(k)$ being the type of k. Define $m_{i,j} = E^i Z_1(j)$, assume $M = (m_{i,j})$ to be positively regular in the usual sense and let ρ be the Frobenius-Perron root of M with associated left and right eigenvectors v, u. We consider throughout the supercritical case $\rho > 1$, and defining

$$a \cdot b = a(1)b(1) + \cdots + a(p)b(p), \qquad |a| = |a(1)| + \cdots + |a(p)|,$$

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for p-vectors a, b, we normalize by $v \cdot u = 1$, $|v| = 1$. Since $E(Z_{n+1} | \mathfrak{F}_n) = Z_n M$, the relation $Mu = \rho u$ implies that $\{W_n\} = \{\rho^{-n} Z_n \cdot u\}$ is a nonnegative martingale. Defining $W = \lim_n W_n$, it is well known that([1])

$$\lim_n \rho^{-n} Z_n = Wv \tag{1.1}$$

and that $\{W > 0\}$ coincides with the set $\{Z_n \neq 0 \text{ for all } n\}$ of nonextinction under mild moment conditions. In fact, our basic assumption

$$E^i |Z_1|^2 < \infty, \qquad i = 1, \ldots, p, \tag{1.2}$$

is more than sufficient for this.

The problem with which we are concerned is this. Given any p-vector a such that $v \cdot a = 0$, we want to describe the asymptotic behavior of the linear functional $Z_n \cdot a$ in a manner more precise than the estimate $Z_n \cdot a = o(\rho^n)$ provided by (1.1). In order to indicate the results of the literature (Kesten and Stigum [12]; Athreya [3], [4], [5]) and state our own, we must introduce the Jordan canonical form of M. We have here given a set of (possibly complex) vectors $u_{\nu,j}$; $\nu = 1, \ldots, \bar{\nu}, j = 1, \ldots, \bar{j}(\nu)$ and to each ν an eigenvalue ρ_ν, such that

$$Mu_{\nu,1} = \rho_\nu u_{\nu,1}, \quad Mu_{\nu,j} = u_{\nu,j-1} + \rho_\nu u_{\nu,j}, \qquad j = 2, \ldots, \bar{j}(\nu). \tag{1.3}$$

Furthermore, each (complex) vector a has a unique expansion in the $u_{\nu,j}$,

$$a = \sum_{\nu=1}^{\bar{\nu}} \sum_{j=1}^{\bar{j}(\nu)} u^*_{\nu,j}[a] u_{\nu,j}, \tag{1.4}$$

and the asymptotic behavior of $M^r a$ follows then from

$$M^r u_{\nu,j} = \sum_{i=1}^{j} \rho_\nu^{r-j+i} \binom{r}{j-i} u_{\nu,i}. \tag{1.5}$$

The known results on the limiting behavior of $Z_n \cdot a$ (a real) are in terms of

$$\lambda = \lambda(a) = \sup\{|\rho_\nu| \colon u^*_{\nu,j}[a] \neq 0 \text{ for some } j\},$$

$$\gamma = \gamma(a) = \sup\{j \colon u^*_{\nu,j}[a] \neq 0 \text{ for some } \nu \text{ with } |\rho_\nu| = \lambda(a)\}.$$

If $\lambda^2 > \rho$, one can exhibit a sequence $\{H_n\}$ of r.v. such that

$$\lim_n \{\lambda^{-n} n^{-(\gamma-1)} Z_n \cdot a - H_n\} = 0. \tag{1.6}$$

If $\lambda^2 \leqslant \rho$, it is easily seen that σ^2 given by

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([1]) All relations between random variables (r.v.) are understood to hold almost surely.

$$\sigma^2 = \lim_n \frac{v \cdot Var^{*} Z_n \cdot a}{\rho^n n^{2\gamma-1}} \quad \text{if } \lambda^2 = \rho, \tag{1.7}$$

$$\sigma^2 = \lim_n \frac{v \cdot Var^{*} Z_n \cdot a}{\rho^n} \quad \text{if } \lambda^2 < \rho$$

exists and that, except for special cases where $\sigma^2 = 0$, $0 < \sigma^2 < \infty$. Then it is known that

$$\lim_n P\left(\frac{Z_n \cdot a}{(\sigma^2 Z_n \cdot u)^{1/2}} \leqslant y | Z_n \neq 0\right) = \Phi(y) \quad \text{if } \lambda^2 < \rho, \tag{1.8}$$

$$\lim_n P\left(\frac{Z_n \cdot a}{\left(\sigma^2 Z_n \cdot u\, n^{2\gamma-1}\right)^{1/2}} \leqslant y | Z_n \neq 0\right) = \Phi(y) \quad \text{if } \lambda^2 = \rho. \tag{1.9}$$

Here, as usual, $\Phi(y) = (2\pi)^{-1/2} \int_{-\infty}^{y} \exp(-z^2/2)\, dz$. The normalizing constant $Z_n \cdot u$ behaves, of course, like $\rho^n W$.

In (1.6), the explicit expressions for the H_n show that $\overline{\lim}_n |H_n| < \infty$ always, and it can also be seen that, except for special cases, $\overline{\lim}_n |H_n| \neq 0$. Thus the question of a.s. behavior is essentially settled by (1.6) if $\lambda^2 > \rho$, while the information provided by (1.8), (1.9) is much more limited. The complete answer is here provided by our main result:

THEOREM 1. *Suppose* $\lambda^2 \leqslant \rho$, $\sigma^2 > 0$ *and let*

$$C_n = \begin{cases} (2\sigma^2 Z_n \cdot u \log n)^{1/2} & \text{if } \lambda^2 < \rho, \\ \left(2\sigma^2 Z_n \cdot u\, n^{2\gamma-1} \log \log n\right)^{1/2} & \text{if } \lambda^2 = \rho. \end{cases}$$

Then on $\{W > 0\}$,

$$\overline{\lim_n} \frac{Z_n \cdot a}{C_n} = 1, \quad \underline{\lim_n} \frac{Z_n \cdot a}{C_n} = -1.$$

As will be seen, not only are the normalizing constants not the same when $\lambda^2 < \rho$ and $\lambda^2 = \rho$, but the proofs are also entirely different. The case $\lambda^2 = \rho$ is, as in the proofs of the central limit theorems in the literature, the more complicated. In the case $\lambda^2 < \rho$ Theorem 1 is derived with relative ease from

PROPOSITION 1. *Let* $Y = Y(Z_0, Z_1, \ldots)$ *be some functional of the process such that* $E^i Y = 0$, $i = 1, \ldots, p$, $0 < \sigma^2 = v \cdot \operatorname{Var} Y < \infty$, *and let* Y_k *be the corresponding functional of the line of descent initiated by* $k \in \mathcal{G}_n$. *Then on* $\{W > 0\}$,

$$\overline{\lim_n} \left(\sum_{k \in \mathcal{G}_n} Y_k \Big/ C_n \right) \leqslant 1$$

where $C_n = (2\sigma^2 Z_n \cdot u \log n)^{1/2}$, *with the inequality replaced by equality if* Y *is* $\mathfrak{F}_r$*-measurable for some* $r < \infty$.

This result has other applications as well. For example, Heyde's iterated logarithm law for the one-type Galton-Watson process [10] and, more generally, results on the rate of convergence in (1.6) can easily be derived. We discuss this in more detail in §4, where some further generalizations are also studied, namely to continuous time and more general processes, like branching diffusions. Also some a.s. limit statements along the lines of the author [1] are given for infinite variance.

2. **Proofs of Proposition 1 and Theorem 1, $\lambda^2 < \rho$.** In order to avoid making trivial exceptions on the set of extinction, we assume from now on that $P(W > 0) = 1$. Also, the proofs of the $\overline{\lim}$ and the $\underline{\lim}$ parts of the results are always similar and we treat only $\overline{\lim}$. In contrast, the proofs of $\overline{\lim} \leqslant 1$ and $\overline{\lim} \geqslant 1$ are certainly not the same. The proof of Proposition 1 is based upon normal approximations and the elementary

LEMMA 1. *Let* $\{T_n\}$ *be a sequence of random variables such that*

$$\sum_{n=0}^{\infty} \Delta_n < \infty, \tag{2.1}$$

where $\Delta_n = \sup_{-\infty<y<\infty} |P(T_n \leqslant y|\mathfrak{F}_n) - \Phi(y)|$. *Then*

$$\overline{\lim_n} \frac{T_n}{(2\log n)^{1/2}} \leqslant 1, \tag{2.2}$$

with the inequality replaced by equality if T_n *is* $\mathfrak{F}_{n+r}$*-measurable for some* $r < \infty$.

PROOF. It is well known that $1 - \Phi(y) \simeq (2\pi)^{-1/2}\exp(-y^2/2)/y$ as $y \to \infty$(2). Therefore for $\gamma > 1$,

$$\sum_{n=0}^{\infty} P\left(T_n > (2\gamma \log n)^{1/2}|\mathfrak{F}_n\right) \leqslant \sum_{n=0}^{\infty} \left\{1 - \Phi\left((2\gamma \log n)^{1/2}\right) + \Delta_n\right\}$$

$$= \sum_{n=0}^{\infty} \left\{O\left(\frac{1}{n^{\gamma}(\log n)^{1/2}}\right) + \Delta_n\right\} < \infty,$$

and the conditional Borel-Cantelli lemma gives $\overline{\lim}\, T_n/(2\log n)^{1/2} \leqslant \gamma^{1/2}$. The fact that one need not require T_n to be $\mathfrak{F}_{n+1}$-measurable is not in most standard textbooks and we refer to Meyer [15, p. 9]. As $\gamma \to 1$, (2.2) follows. In the same way we get for $\gamma < 1$,

(2) $\simeq$ means throughout that the ratio is one in the limit.

$$\sum_{n=0}^{\infty} P(T_n > (2\gamma \log n)^{1/2} | \mathfrak{F}_n) \geqslant \sum_{n=0}^{\infty} \{1 - \Phi((2\gamma \log n)^{1/2}) - \Delta_n\}$$
$$= \sum_{n=0}^{\infty} \left\{ O\left(\frac{1}{n^{\gamma}(\log n)^{1/2}}\right) - \Delta_n \right\} = \infty.$$

If T_n is $\mathfrak{F}_{n+r}$-measurable, this implies $\overline{\lim}_n T_n/(2 \log n)^{1/2} \geqslant \gamma^{1/2}$. For this, it suffices to refer to the standard version of the conditional Borel-Cantelli lemma (Neveu [16, p. 151]). Now let $\gamma \to 1$. □

PROOF OF PROPOSITION 1. Define(3)

$$Y'_k = Y_k I(|Y_k| \leqslant \rho^{n/2}), \qquad \tilde{Y}_k = Y'_k - EY'_k,$$
$$S_n = \sum_{k \in \mathcal{G}_n} \tilde{Y}_k, \quad \tau_n^2 = \operatorname{Var}(S_n | \mathfrak{F}_n), \quad T_n = S_n/\tau_n.$$

By a standard moment inequality,

$$E|\tilde{Y}_k|^3 \leqslant E|Y'_k|^3 + 3E|Y'_k|(E|Y'_k|)^2 + 3E|Y'_k|EY'^2_k + (E|Y'_k|)^3$$
$$\leqslant 8E|Y'_k|^3 = 8\int_0^{\rho^{n/2}} y^3\, dF^{i(k)}(y),$$

where $i(k)$ is the type of k and $F^i(y) = P^i(|Y| \leqslant y)$. Letting C be the Berry-Esseen constant, we get

$$\Delta_n \leqslant 8C\tau_n^{-3} \sum_{i=1}^{p} Z_n(i) \int_0^{\rho^{n/2}} y^3\, dF^i(y).$$

It is readily checked that

$$\lim_n \frac{\tau_n^2}{\sigma^2 Z_n \cdot u} = 1. \tag{2.3}$$

Therefore $\tau_n^{-3} Z_n(i) = O(\rho^{-n/2})$, and (2.1) follows from

$$\sum_{n=0}^{\infty} \rho^{-n/2} \int_0^{\rho^{n/2}} y^3\, dF^i(y) = \int_0^{\infty} y^3 \sum \rho^{-n/2} I(y \leqslant \rho^{n/2})\, dF^i(y)$$
$$= \int_0^{\infty} y^3 O(y^{-1})\, dF^i(y) = \int_0^{\infty} O(y^2)\, dF^i(y) < \infty.$$

Thus (2.2) holds and it only remains to prove

$$\overline{\lim_n}\left(\sum_{k \in \mathcal{G}_n} Y_k \Big/ C_n\right) = \overline{\lim_n} \frac{T_n}{(2 \log n)^{1/2}}.$$

(3) EY'_k etc. is a convenient notation, but strictly speaking we mean $E(Y'_k | \mathfrak{F}_n)$.

Recalling (2.3) and the explicit definitions of Y'_k, $\tilde{Y}_k$, T_n, it suffices that

$$\sum_{k \in \mathcal{I}_n} \{Y_k - Y'_k\} = o\left(\rho^{n/2}(\log n)^{1/2}\right), \quad \sum_{k \in \mathcal{I}_n} EY'_k = o(\rho^{n/2}(\log n)^{1/2})$$

or, appealing to Kronecker's lemma, that

$$\text{(2.4)} \qquad \begin{aligned} &\sum_{n=0}^{\infty} \rho^{-n/2}(\log n)^{-1/2} \sum_{k \in \mathcal{I}_n} |Y_k - Y'_k| < \infty, \\ &\sum_{n=0}^{\infty} \rho^{-n/2}(\log n)^{-1/2} \sum_{k \in \mathcal{I}_n} |EY'_k| < \infty. \end{aligned}$$

Noting that $|Z_n(i)| = O(\rho^n)$ and that

$$\begin{aligned} |EY'_k| &= |E(Y'_k - Y_k)| \leqslant E|Y'_k - Y_k| = E|Y_k| I(|Y_k| > \rho^{n/2}) \\ &= \int_{\rho^{n/2}}^{\infty} y \, dF^{i(k)}(y), \end{aligned}$$

it suffices for both assertions of (2.4) that

$$\text{(2.5)} \qquad \sum_{n=0}^{\infty} \rho^{-n/2}(\log n)^{-1/2} \rho^n \int_{\rho^{n/2}}^{\infty} y \, dF^i(y) < \infty, \qquad i = 1, \ldots, p$$

(for the first, take the mean). And (2.5) certainly holds since even

$$\begin{aligned} \sum_{n=0}^{\infty} \rho^{n/2} \int_{\rho^{n/2}}^{\infty} y \, dF^i(y) &= \int_0^{\infty} y \sum \rho^{n/2} I(y > \rho^{n/2}) \, dF^i(y) \\ &= \int_0^{\infty} O(y^2) \, dF^i(y) < \infty. \quad \square \end{aligned}$$

Theorem 1 is easily derived from Proposition 1, once we have shown

LEMMA 2. *Define* $A_r = \sup_N |Z_{N+1} \cdot M^r a| / C_{N+1}$. *Then* $A_r < \infty$. *More precisely,* $A_r = O(\lambda^r r^{\gamma-1})$ *as* $r \to \infty$.

PROOF. Let $b_1, \ldots, b_s$ be the set of all vectors of the form $b_i = \operatorname{Re} u_{\nu,j}$ or $b_i = \operatorname{Im} u_{\nu j}$ with $|\rho_\nu| \leqslant \lambda, j \leqslant \gamma$. Writing (1.4), (1.5) in real form, we have an expansion $M^r a = \alpha_1^r b_1 + \cdots + \alpha_s^r b_s$, where $\alpha^r = |\alpha_1^r| + \cdots + |\alpha_s^r|$ satisfies

$$\text{(2.6)} \qquad \begin{aligned} &\alpha^r = O(\lambda^r r^{\gamma-1}) \quad \textit{as } r \to \infty, \quad \textit{in particular,} \\ &\sum_{n=0}^{\infty} \frac{\alpha^{n+r}}{\rho^{n/2}} = O(\lambda^r r^{\gamma-1}). \end{aligned}$$

In the definition of A_r, we can replace C_{N+1} with

$$c_{N+1} = \left(\rho^{N+1} \log(N+1)\right)^{1/2},$$

since $\sup_N c_{N+1}/C_{N+1} < \infty$. We use the expansion

(2.7)

$$Z_{N+1} \cdot M^r a = Z_0 \cdot M^{N+r+1} a + \sum_{n=0}^{N} \{Z_{n+1} \cdot M^{N+r-n} a - Z_n \cdot M^{N+r-n+1} a\}.$$

Define $B_i = \sup_n |Z_{n+1} \cdot b_i - Z_n \cdot Mb_i|/c_n$. Then $B_i < \infty$, trivially if $v \cdot \mathrm{Var}^{\bullet} Z_1 \cdot b_i = 0$, and otherwise by Proposition 1 with

$$Y = Z_1 \cdot b_i - Z_0 \cdot Mb_i, \quad \sum_{k \in \mathcal{G}_n} Y_k = Z_{n+1} \cdot b_i - Z_n \cdot Mb_i.$$

Thus also $B = \max\{B_1, \ldots, B_s\} < \infty$ and inserting in (2.7) yields

$$\frac{|Z_{N+1} \cdot M^r a|}{c_{N+1}} \leqslant \frac{Z_0 \cdot M^{N+r+1} a}{c_{N+1}} + B \sum_{n=0}^{N} \alpha^{N+r-n} c_n \Big/ c_{N+1}$$

$$\leqslant O\left(\frac{\alpha^{N+r+1}}{c_{N+1}}\right) + B \sum_{n=0}^{N} \frac{\alpha^{n+r}}{\rho^{n/2}}.$$

(2.6) completes the proof. □

PROOF OF THEOREM 1, $\lambda^2 < \rho$. Define $\sigma_r^2 = v \cdot \mathrm{Var}^{\bullet} Z_r \cdot a/\rho^r$,

$$\beta_{n,r} = \frac{Z_{n+r} \cdot a - Z_n \cdot M^r a}{(2\rho^r \sigma_r^2 Z_n \cdot u \log n)^{1/2}}, \quad \gamma_{n,r} = \left(\frac{\rho^r Z_n \cdot u \log n}{Z_{n+r} \cdot u \log(n+r)}\right)^{1/2}.$$

Then for fixed r, $\lim_n \gamma_{n,r} = 1$ and, applying Proposition 1 in a similar manner as in the proof of Lemma 2, $\overline{\lim}_n \beta_{n,r} = 1$. Now

$$Z_{n+r} \cdot a/C_{n+r} \leqslant (\sigma_r/\sigma)\gamma_{n,r}\beta_{n,r} + \rho^{-r/2} A_r \gamma_{n,r},$$

$$\overline{\lim_n} \frac{Z_n \cdot a}{C_n} \leqslant \sigma_r/\sigma + \rho^{-r/2} A_r \to 1 + 0 \quad \text{as } r \to \infty.$$

$\underline{\lim} \geqslant 1$ is similar from

$$Z_{n+r} \cdot a/C_{n+r} \geqslant (\sigma_r/\sigma)\gamma_{n,r}\beta_{n,r} - \rho^{-r/2} A_r \gamma_{n,r}. \quad \square$$

3. **Proof of Theorem 1, $\lambda^2 = \rho$.** The first step is to show

LEMMA 3. *For any sequence $\{t(i)\}$ such that $\lim_i t(i)/\theta^i = 1$, $1 < \theta < \infty$, we have*

(3.1)
$$\overline{\lim_i} \frac{Z_{t(i)} \cdot a}{C_{t(i)}} \leqslant 1,$$

$$\overline{\lim_i} \frac{Z_{t(i)} \cdot a - Z_{t(i-1)} \cdot M^{t(i)-t(i-1)}a}{C_{t(i)}} \geqslant \left(1 - \frac{1}{\theta}\right)^{\gamma-1/2}. \tag{3.2}$$

The proof is, as in §2, based upon normal approximations and Lemma 1. To estimate the remainder term in the central limit theorem (1.9), we fix N, rewrite the expansion (2.7) (with $r = 0$) of $Z_{N+1} \cdot a$ in a sum of martingale increments as

$$Z_{N+1} \cdot a = Z_0 \cdot M^{N+1}a + \sum_{n=0}^{N} \sum_{k \in \mathcal{G}_n} (U_k - EU_k) \cdot M^{N-n}a, \tag{3.3}$$

and apply the central limit theory for martingales to the martingale (3.3) indexed by the total set of particles k rather than the one indexed by the generation number n as in (2.7). We remark in this connection that this approach leads to proofs of (1.9) which seem more striaghtforward than those of the literature, but we shall not give the details here.

Whenever ρ_ν, ρ_μ are complex conjugates, $\rho_\nu = \bar{\rho}_\mu$, we can assume $u_{\nu,j} = \bar{u}_{\mu,j}, j = 1, \ldots, \bar{j}(\nu) = \bar{j}(\mu)$. Define $\mathcal{V}_j$ as the complex span of the $u_{\nu,j}$ with $|\rho_\nu|^2 = \rho$, $\mathcal{L}_j$ as the real span of the $\operatorname{Re} u_{\nu,j}$, $\operatorname{Im} u_{\nu,j}$ for the same ν and let $\mathcal{V} = \mathcal{V}_1 + \cdots + \mathcal{V}_\gamma$, $\mathcal{L} = \mathcal{L}_1 + \cdots + \mathcal{L}_\gamma$. Clearly, $\mathcal{L}_j = \mathcal{V}_j \cap \mathbf{R}^p$ and $\lambda(b)^2 = \rho$, $\gamma(b) = j$ when $b \in \mathcal{L}_j$. From §2, we can, without loss of generality, assume $u^*_{\nu,j}[a] = 0$ when $|\rho_\nu|^2 < \rho$, i.e. $a \in \mathcal{L}$. Since M is one-one on $\mathcal{V}$, $M^{-r}\colon \mathcal{V} \to \mathcal{V}$ exists for each $r = 0, 1, 2, \ldots$ and clearly $M^{-r}\mathcal{L} = \mathcal{L}$. One can check that (1.5) remains valid for $r = -1, -2, \ldots$ and $|\rho_\nu|^2 = \rho$ and one easily gets

LEMMA 4. *There exist $b_{r,j} \in \mathcal{L}_j, j = 1, \ldots, \gamma, r = 0, \pm 1, \pm 2, \ldots$, such that*

$$M^r a = \rho^{r/2} \sum_{j=1}^{\gamma} |r|^{\gamma-j} b_{r,j}, \qquad r = 0, \pm 1, \pm 2, \ldots.$$

Furthermore, for each j, $\{b_{n,j}\}$ is relatively compact.

Combining with (1.7), we have

LEMMA 5. *As $n, r, \to \infty$,*

$$\left.\begin{aligned} |E^i Z_n \cdot M^r a| &= O(\rho^{(n+r)/2}(nr)^{\gamma-1}), \\ \operatorname{Var}^i Z_n \cdot M^r a &= O(\rho^{n+r} n^{2\gamma-1} r^{2\gamma-2}), \end{aligned}\right\} \quad i = 1, \ldots, p.$$

To obtain the estimate corresponding to (2.1), we use a classical result of Levy [14, p. 243]. [14] does not apply directly to (3.1); some modifications are needed. We let $\tau_n = |Z_0| + \cdots + |Z_n|$ and assume the nth generation represented as the $k \in \mathbf{N}$ such that $\tau_{n-1} < k \leqslant \tau_n$. This inequality is always assumed valid whenever k and n vary together. For (3.2), let $\underline{n}(i) = t(i-1)$, while for (3.1) we choose $\underline{n}(i)$ such that, defining $\Delta(i) = t(i) - \underline{n}(i)$, we have

$$\text{(3.4)} \qquad \underline{n}(i) \uparrow \infty, \quad \lim_i \frac{\underline{n}(i)}{t(i)} = 0, \quad \sum_{i=1}^{\infty} E \frac{|Z_{\underline{n}(i)} \cdot M^{\Delta(i)} a|^2}{\rho^{t(i)} t(i)^{2\gamma-1}} < \infty.$$

This is possible by Lemma 5. Further, suppose we have defined on our probability space r.v. $B_{k,i}$, which are mutually independent and independent of the branching process, with $P(B_{k,i} = 1) = P(B_{k,i} = -1) = \frac{1}{2}$ and let

$$\tilde{U}_k = U_k I(|U_k| \leqslant \rho^{n/2}), \qquad D_i = (\sigma^2 W_{\underline{n}(i)} \rho^{t(i)} t(i)^{2\gamma-1})^{1/2},$$

$$Y_{k,i} = \begin{cases} (\tilde{U}_k - E\tilde{U}_k) \cdot M^{t(i)-n-1} a / D_i, & \underline{k}(i) = \tau_{\underline{n}(i)} + 1 \leqslant k \leqslant \tau_{t(i)}, \\ B_{k,i}/D_i, & k > \tau_{t(i)}. \end{cases}$$

$$\mathfrak{G}_k = \sigma(U_j, j \leqslant k), \qquad X_{k,i} = \sum_{j=\underline{k}(i)}^{k} Y_{j,i}, \quad s_{k,i}^2 = \sum_{j=\underline{k}(i)}^{k} E(Y_{j,i}^2 | \mathfrak{G}_{j-1}).$$

LEMMA 6.

$$\overline{\lim_i} \frac{X_{\tau_{t(i)},i}}{(2 \log i)^{1/2}} = \overline{\lim_i} \frac{Z_{t(i)} \cdot a - Z_{\underline{n}(i)} \cdot M^{\Delta(i)}}{C_{t(i)}}.$$

PROOF. Let $V_k = U_k - EU_k - (\tilde{U}_k - E\tilde{U}_k)$. Clearly, $D_i(2 \log i)^{1/2} \simeq C_{t(i)}$ so that it suffices to show

$$\left| \sum_{k=\underline{k}(i)}^{\tau_{t(i)}} V_k \cdot M^{t(i)-n-1} a \right| = o((\rho^{t(i)} t(i)^{2\gamma-1} \log i)^{1/2}),$$

which by Lemma 4 is weaker than

$$\sum_{k=\underline{k}(i)}^{\tau_{t(i)}} \rho^{(t(i)-n-1)/2} (t(i) - n - 1)^{\gamma-1} |V_k| = o((\rho^{t(i)} t(i)^{2\gamma-1})^{1/2}),$$

$$\sum_{k=\underline{k}(i)}^{\tau_{t(i)}} \rho^{-n/2} |V_k| = o(t(i)^{1/2}).$$

But by the last argument in the proof of Proposition 1, even $\sum_0^\infty \rho^{-n/2} |V_k| < \infty$. $\square$

We next need some elementary facts concerning second moments. Define for $n, m = 0, 1, 2, \ldots,$

$$c_n = \rho^{-n/2} n^{-(\gamma-1)} M^n a, \quad \sigma_n^2(i) = \mathrm{Var}^i Z_1 \cdot c_n, \quad \underline{\sigma}_n^2 = v \cdot \sigma_n^2,$$

$$\sigma_{n,m}^2(i) = \mathrm{Var}^i Z_1 \cdot c_n I(|Z_1| \leqslant \rho^{m/2}).$$

Then $\{c_n\}$, $\{\sigma_n^2(i)\}$, etc. are relatively compact and one easily checks

$$\lim_n \sup_m |\rho^{-n} Z_n \cdot \sigma_{m,n}^2 - W \underline{\sigma}_m^2| = 0, \tag{3.5}$$

$$\sigma^2 = \rho^{-1} \lim_N \left(\sum_{n=0}^{N} n^{2\gamma-2} \underline{\sigma}_n^2 \right) \Big/ (N+1)^{2\gamma-1} \quad \text{(cf. (1.7),(2.7))}, \tag{3.6}$$

$$\begin{aligned} s_{\tau_{t(i)},i}^2 &= \sum_{n=\underline{n}(i)}^{t(i)-1} \rho^{t(i)-n-1} (t(i)-n-1)^{2\gamma-2} Z_n \cdot \sigma_{t(i)-n-1,n}^2 \Big/ D_i^2 \\ &\simeq W \rho^{t(i)-1} \sum_{j=0}^{t(i)-\underline{n}(i)-1} j^{2\gamma-2} \underline{\sigma}_j^2 \Big/ D_i^2 \simeq \left(1 - \frac{\underline{n}(i)}{t(i)}\right)^{2\gamma-1}, \end{aligned} \tag{3.7}$$

using (3.6) for the last estimate in (3.7) and (3.5) for the preceding one.

Inspection of the bounds on the $Y_{k,i}$ (i fixed) and reference to [14] yields

LEMMA 7. *Let* $0 < \kappa < \infty$ *and define* $\overline{k}(i) = \inf\{k \geqslant \underline{k}(i) \colon s_{k,i}^2 \geqslant \kappa^2\}$. *Then for all* y,

$$|P(X_{\overline{k}(i),i}^* \leqslant y\kappa | \mathfrak{F}_{\underline{n}(i)}) - \Phi(y)| \leqslant 6\varepsilon_i^{1/4},$$

where $X_{\overline{k}(i),i}^* = X_{\overline{k}(i)-1,i} + c_i Y_{\overline{k}(i),i}$ *for some* $\mathfrak{G}_{\overline{k}(i)-1}$*-measurable* c_i, $0 \leqslant c_i \leqslant 1$, $\varepsilon_i = D/(\kappa^2 W_{\underline{n}(i)} t(i))^{1/2}$ *for some constant* D. *In particular,* $\sum \varepsilon_i^{1/4} < \infty$.

PROOF OF (3.1). Let $\delta > \kappa > 1$. Then

$$\begin{aligned} &\sum_{i=1}^{\infty} P(X_{\tau_{t(i)},i} > \delta(2\log i)^{1/2}, \overline{k}(i) > \tau_{t(i)} | \mathfrak{F}_{\underline{n}(i)}) \\ &\leqslant 2 \sum_{i=1}^{\infty} P(X_{\tau_{t(i)},i} > \delta(2\log i)^{1/2}, \overline{k}(i) > \tau_{t(i)}, \\ &\qquad B_{\tau_{t(i)}+1,i} + \cdots + B_{\overline{k}(i)-1,i} + c_i B_{\overline{k}(i),i} \geqslant 0 | \mathfrak{F}_{\underline{n}(i)}) \\ &\leqslant 2 \sum_{i=1}^{\infty} P(X_{\overline{k}(i),i}^* > \delta(2\log i)^{1/2} | \mathfrak{F}_{\underline{n}(i)}) < \infty, \end{aligned}$$

where the last inequality is similar to the proof of Lemma 1, replacing (2.1) by Lemma 7. Since $\kappa^2 > 1$ and (3.4) and (3.7) ensure $\overline{k}(i) > \tau_{t(i)}$ eventually, it follows that

$$\overline{\lim_i} \frac{X_{\tau_{t(i)},i}}{(2\log i)^{1/2}} \leqslant \delta.$$

As $\delta \downarrow 1$, (3.1) follows from Lemma 6 and (3.4). $\square$

PROOF OF (3.2). Here $\underline{n}(i) = t(i-1)$ and thus the r.h.s. of (3.7) $\simeq (1-\theta^{-1})^{2\gamma-1}$. Taking $0 < \kappa^2 < (1-\theta^{-1})^{2\gamma-1}$ will thus ensure $\tau_{t(i-1)} < \bar{k}(i) < \tau_{t(i)}$ eventually. Let

$$A_{k,i} = E((X_{\tau_{t(i)},i} - X_{k,i})^2 | \mathfrak{G}_k) = E(s^2_{\tau_{t(i)},i} - s^2_{k,i} | \mathfrak{G}_k),$$

$$M_i = \max_{\tau_{t(i-1)} < k \leqslant \tau_{t(i)}} (X_{k,i} - (2A_{k,i})^{1/2}).$$

In the same way as in (3.7) one can check that $A_{k,i} = O(1)$, and it follows that

$$\text{(3.8)} \qquad \overline{\lim_i} \frac{M_i}{(2\log i)^{1/2}} = \overline{\lim_i} \max_{\tau_{t(i-1)} < k \leqslant \tau_{t(i)}} \frac{X_{k,i}}{(2\log i)^{1/2}}$$
$$\geqslant \overline{\lim_i} \frac{X_{\bar{k}(i),i}}{(2\log i)^{1/2}} = \overline{\lim_i} \frac{X^*_{\bar{k}(i),i}}{(2\log i)^{1/2}} \geqslant \kappa$$

(the last inequality as in §2 from Lemma 7). By the martingale version of the Kolmogorov-Levy inequality (see, e.g., pp. 286–287 of Stout [17]),

$$\sum_{i=1}^{\infty} P\left(\frac{X_{\tau_{t(i)},i}}{(2\log i)^{1/2}} > \kappa - \varepsilon \Big| \mathfrak{F}_{\underline{n}(i)}\right) \geqslant \frac{1}{2} \sum_{i=1}^{\infty} P\left(\frac{M_i}{(2\log i)^{1/2}} > \kappa - \varepsilon \Big| \mathfrak{F}_{\underline{n}(i)}\right)$$
$$= \infty,$$

$$\overline{\lim_i} \frac{X_{\tau_{t(i)},i}}{(2\log i)^{1/2}} \geqslant \kappa - \varepsilon.$$

Here we have used the conditional Borel-Cantelli lemma both ways and (3.8). Now let $\varepsilon \downarrow 0$, $\kappa^2 \uparrow (1-\theta^{-1})^{2\gamma-1}$ and use Lemma 6. $\square$

LEMMA 8. *Let $j = 1, \ldots, \gamma$ and let $\{t(i)\}$ be as in Lemma 3. Suppose $B \subseteq \mathfrak{L}_j$ is relatively compact and define*

$$\sigma^2(b) = \lim_n \frac{v \cdot \mathrm{Var}^{\boldsymbol{\cdot}} Z_n \cdot b}{\rho^n n^{2\gamma-1}}, \quad D_n = \left(2 \sup_{b\in B} \sigma^2(b) Z_n \cdot u\, n^{2j-1} \log\log n\right)^{1/2}.$$

Then $\overline{\lim}_i \sup_{b\in B} |Z_{t(i)} \cdot b| / D_{t(i)} \leqslant 1$.

PROOF. Choose $b_1, \ldots, b_m \in \mathfrak{L}_j$ as a basis for $\mathfrak{L}_j$ and, to a given $\varepsilon > 0$, N and $b^1, \ldots, b^N \in B$ such that to any $b \in B$ there exist $s(b)$ and $c_r(b)$ such that $b = b^{s(b)} + \sum_1^m c_r(b)\, b_r$, $\Sigma |c_r(b)| < \varepsilon$. Then the $\overline{\lim}$ is bounded by

$$\overline{\lim_i}\left(\max_{s=1,\ldots,N}|Z_{t(i)}\cdot b^s|+\varepsilon\max_{r=1,\ldots,m}|Z_{t(i)}\cdot b_r|\right)\Big/D_{t(i)}$$

which cannot exceed $1+\varepsilon\max\{\sigma(b_1),\ldots,\sigma(b_m)\}/\sup_{b\in B}\sigma(b)$ by (3.1). □

LEMMA 9. *Let a, $\gamma=\gamma(a)$, $\sigma^2=\sigma^2(a)$, $\{t(i)\}$ be as in Lemma 3 and let, for each i, $R(i)$ be a subset of integers. Then*

$$\overline{\lim_i}\sup_{r\in R(i)}\frac{|Z_{t(i)}\cdot M^r a|}{\rho^{r/2}C_{t(i)}}\leqslant 1+\sum_{j=1}^{\gamma-1}w_j\mu^{\gamma-j}, \tag{3.9}$$

where $w_j^2=\sup_r\sigma^2(b_{r,j})/\sigma^2<\infty$, $\mu=\overline{\lim}_i\sup_{r\in R(i)}|r|/t(i)$.

PROOF. Expanding $M^r a$ as in Lemma 4, the r.h.s. of (3.9) is bounded by

$$\sum_{j=1}^{\gamma}\overline{\lim_i}\sup_{r\in R(i)}\left(\frac{|r|}{t(i)}\right)^{\gamma-j}\frac{Z_{t(i)}\cdot b_{r,j}}{(2\sigma^2W\rho^{t(i)}t(i)^{2j-1}\log\log t(i))^{1/2}},$$

and appealing to Lemma 8, we need only remark that $\sigma^2(b_{r,\gamma})=\sigma^2(\rho^{-r/2}M^r a)=\sigma^2(a)=\sigma^2$, using (3.6). □

We can now finally complete the

PROOF OF THEOREM 1, $\lambda^2=\rho$. $\overline{\lim}\geqslant 1$ is easily derived from (3.2) and Lemma 9 with $R(i)=\{\Delta(i+1)\}$. Remarking that

$$C_{t(i)}\simeq\theta^{\gamma-1/2}\rho^{\Delta(i)/2}C_{t(i-1)},\qquad \mu=\theta-1,$$

we get

$$\begin{aligned}\overline{\lim_n}\frac{Z_n\cdot a}{C_n}&\geqslant\overline{\lim_i}\frac{Z_{t(i)}\cdot a}{C_{t(i)}}\\&\geqslant\left(1-\frac{1}{\theta}\right)^{2\gamma-1}-\frac{1}{\theta^{\gamma-1/2}}\left(1+\sum_{j=1}^{\gamma-1}w_j(\theta-1)^{\gamma-j}\right).\end{aligned}$$

Let $\theta\uparrow\infty$.

For $\overline{\lim}\leqslant 1$, it remains to bound the $Z_n\cdot a$ with $t(i-1)\leqslant n\leqslant t(i)$. We approximate $Z_n\cdot a$ by $Z_{t(i)}\cdot M^{n-t(i)}a$. More precisely, a straightforward modification of the argument on pp. 286–287 of [17] yields the inequality $P(M''_i>\varepsilon)\leqslant 2P(M'_i>\varepsilon)$, where

$$R(i)=\{0,-1,\ldots,t(i-1)-t(i)\},\quad M'_i=\sup_{r\in R(i)}\frac{Z_{t(i)}\cdot M^r a}{\rho^{r/2}\rho^{t(i)/2}},$$

$$A_n=\operatorname{Var}\left(\frac{Z_{t(i)}\cdot M^{n-t(i)}a}{\rho^{n/2}}\,\middle|\,\mathfrak{F}_n\right),\quad M''_i=\max_{t(i-1)\leqslant n\leqslant t(i)}\left[\frac{Z_n\cdot a}{\rho^{n/2}}-(2A_n)^{1/2}\right].$$

Letting $E_i=(2\sigma^2W_{t(i-1)}t(i)^{2\gamma-1}\log i)^{1/2}$, it follows that

$$\overline{\lim_i} \frac{M''_i}{E_i} \leqslant \overline{\lim_i} \frac{M'_i}{E_i} \leqslant 1 + \sum_{j=1}^{\gamma-1} w_j\left(1 - \frac{1}{\theta}\right)^{\gamma-j},$$

the r.h. inequality by Lemma 9 and the l.h. by the argument used in the proof of (3.2). By Lemma 5, one can prove that $\lim_n A_n^{1/2}/E_i = 0$ ($t(i-1) \leqslant n \leqslant t(i)$). Thus

$$\overline{\lim_n} \frac{Z_n \cdot a}{C_n} \leqslant \theta^{\gamma-1/2} \overline{\lim_i} \max_{t(i-1) \leqslant n \leqslant t(i)} \frac{Z_n \cdot a}{\rho^{n/2} E_i}$$

$$= \theta^{\gamma-1/2} \overline{\lim_i} \frac{M''_i}{E_i} \leqslant \theta^{\gamma-1/2}\left(1 + \sum_{j=1}^{\gamma-1} w_j\left(1 - \frac{1}{\theta}\right)^{\gamma-j}\right).$$

Let $\theta \downarrow 1$. □

4. **Generalizations.** We first consider the case of continuous time. Instead of the offspring mean matrix M and its iterates, we have a semigroup $\{M_t\}_{t \geqslant 0}$ with infinitesimal generator A, $M_t = e^{At}$. The Jordan canonical form is most naturally expressed in terms of A, but, for simplicity, we define it as earlier relative to M_1. We shall, however, need the fact that the eigenvalues of M_t are of the form $e^{\alpha t}$, with α an eigenvalue of A. Since $e^{\alpha t} \neq 0$, M_t is thus one-one.

THEOREM 2. *Replacing $n \in \mathrm{N}$ with $t \in [0, \infty)$, Theorem 1 remains valid for any supercritical positively regular p-type Markov branching process.*

PROOF. It is clear that the proof for $\lambda^2 = \rho$, more precisely the way to bound $Z_t \cdot a$ when $t(i-1) \leqslant t \leqslant t(i)$, works equally well in continuous time. A similar argument yields $\overline{\lim}_t \leqslant 1$ when $\lambda^2 < \rho$ (which is, of course, the only problem since $\overline{\lim}_t \geqslant \overline{\lim}_n = 1$): Since M_s is one-one for any s and $\{M_s^{-1}a\}_{0 \leqslant s \leqslant \delta}$ is relatively compact, we get

$$\overline{\lim_t} \frac{Z_t \cdot a}{C_t} = \overline{\lim_n} \sup_{(n-1)\delta \leqslant t \leqslant n\delta} \frac{Z_t \cdot a}{C_{n\delta}} \cdot \frac{C_{n\delta}}{C_t}$$

$$\leqslant \overline{\lim_n} \left(\sup_{0 \leqslant s \leqslant \delta} Z_{n\delta} \cdot M_s^{-1}a / C_{n\delta}\rho^{\delta/2}\right) \leqslant \sup_{0 \leqslant s \leqslant \delta} \sigma(M_s^{-1}a)/\sigma(a)\rho^{\delta/2}.$$

As $\delta \downarrow 0$, the r.h.s. tends to one. □

Returning to discrete time, we state some results for infinite variance, more precisely in terms of the condition

$$(4.1) \qquad E^i|Z_1 \cdot a|^{1/\beta} < \infty, \qquad i = 1, \ldots, p \text{ with } \tfrac{1}{2} < \beta < 1.$$

Except for special cases, (4.1) is, of course, equivalent to the condition that the $(1/\beta)$th offspring moment be finite.

THEOREM 3. *Suppose $Ma = \rho_\nu a$ with ρ_ν, a real and $\rho_\nu^2 \leqslant \rho$. Then* (4.1) *is equivalent to $Z_n \cdot a = o(\rho^{n\beta})$.*

THEOREM 4. *Suppose* $Ma = \rho_\nu a$ *with* ρ_ν, *a real and* $\rho < \rho_\nu^2 \leqslant \rho^2$ *and let* $\alpha = \log|\rho_\nu|/\log\rho$. *Then* (4.1) *is equivalent to*: (i) $Z_n \cdot a = o(\rho^{n\beta})$ *if* $\beta > \alpha$, $\rho_\nu \neq \rho$; (ii) *the existence of* $W^* = \lim_n \rho_\nu^{-n} Z_n \cdot a$ *if* $\beta = \alpha$, $\rho_\nu \neq \rho$; (iii) $Z_n \cdot a - \rho_\nu^n W^* = o(\rho^{n\beta})$ *if* $\beta < \alpha$. *Furthermore, if* $0 < \sigma^2 = v \cdot \mathrm{Var}^\bullet W^* < \infty$, *then on* $\{W > 0\}$,

$$\overline{\lim_n} \frac{Z_n \cdot a - \rho_\nu^n W^*}{(2\sigma^2 Z_n \cdot u \log n)^{1/2}} = 1, \quad \underline{\lim_n} \frac{Z_n \cdot a - \rho_\nu^n W^*}{(2\sigma^2 Z_n \cdot u \log n)^{1/2}} = -1. \tag{4.2}$$

The proofs of Theorem 3 and (i), (ii), (iii) of Theorem 4 are in the spirit of the author's paper [1] and the proof of the Marcinkiewicz-Zygmund law of large numbers as given in Neveu [16, pp. 152–155]. Some modifications are needed, but we omit the details. Part (ii) of Theorem 4 was noted at least implicitly by Kesten and Stigum [12]. If $\rho_\nu = \rho$, $W^* = W$, (4.2) may be seen as the multitype analogue of a result of Heyde ([10]; also see Heyde and Leslie [11] and Leslie [13]). (4.2) is also similar in form to iterated logarithm laws for tail sums of independent r.v. (see Barbour [7] and Chow and Teicher [8]). We give the

PROOF OF (4.2). Define $\sigma_r^2 = v \cdot \mathrm{Var}^\bullet \rho_\nu^{-r} Z_r \cdot a$,

$$\alpha_n = \frac{Z_n \cdot a - \rho_\nu^n W^*}{(2\sigma^2 Z_n \cdot u \log n)^{1/2}}, \quad \beta_{n,r} = \frac{\rho_\nu^{-r} Z_{n+r} \cdot a - Z_n \cdot a}{(2\sigma_r^2 Z_n \cdot u \log n)^{1/2}},$$

$$\gamma_{n,r} = \left(\frac{\rho^r Z_n \cdot u \log n}{Z_{n+r} \cdot u \log(n+r)} \right)^{1/2}.$$

From Proposition 1 with $Y = W^* - Z_0 \cdot a$, $\lim_n |\alpha_n| \leqslant 1$ and, similarly, $\underline{\lim}_n \beta_{n,r} = -1$. Now write

$$\alpha_n = \frac{(\rho^{r/2}/\rho_\nu^r)\alpha_{n+r}}{\gamma_{n,r}} - \frac{\sigma_r}{\sigma}\beta_{n,r}.$$

Letting $n \to \infty$ yields $\overline{\lim}\, \alpha_n \geqslant -\rho^{r/2}/\rho_\nu^r + \sigma_r/\sigma$. As $r \uparrow \infty$, it follows that $\overline{\lim}\, \alpha_n \geqslant -0 + 1$. □

(4.2) leads naturally to ask for the general behavior of the remainder term $\varepsilon_n = Z_n \cdot a - \lambda^n n^{\gamma-1} H_n$ in (1.6). Let

$$a_1 = \textstyle\sum^{(a)} u_{\nu,j}^*[a] u_{\nu,j}, \quad a_2 = a - a_1, \quad \delta_n = Z_n \cdot a_1 - \lambda^n n^{\gamma-1} H_n,$$

where $\sum^{(a)}$ extend over the ν, j such that $|\rho_\nu| = \lambda, j = \gamma$. Then $\varepsilon_n = \delta_n + Z_n \cdot a_2$ and one can prove that δ_n obeys the law of the iterated logarithm with normalizing constants $C_n = (2\sigma^2 Z_n \cdot u\, n^{2\gamma(a)-2} \log n)^{1/2}$.

Comparing C_n to the magnitude of $Z_n \cdot a_2$, it follows that the behavior of ε_n is that of $Z_n \cdot a_2$ if either $\lambda(a_2)^2 > \rho$, in which case we have a limit theorem of type (1.6), or if $\lambda(a_2)^2 = \rho$, $\gamma(a_2) \geqslant \gamma(a)$, where we get an iterated logarithm law with normalizing constants $(2\sigma^2 Z_n \cdot u\, n^{2\gamma(a_2)-1} \log\log n)^{1/2}$. Otherwise ε_n behaves like δ_n and we get an iterated logarithm law with normalizing constants C_n.

Finally, we would like to mention that all probabilistic considerations of the present paper carry over essentially unchanged to more general processes like branching diffusions; see, e.g., Asmussen and Hering [2]. Some problems come up, however, in the algebra; e.g., in some of the proofs we have expanded in a finite set of basis vectors and also the whole question of the existence and properties of a suitable Jordan canonical form is more involved. For a simple example of the spectral properties of branching diffusions, we refer to [2], and for a more comprehensive treatment, to Hering [9].

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INSTITUTE OF MATHEMATICAL STATISTICS, UNIVERSITY OF COPENHAGEN, 5 UNIVERSITETSPARKEN, DK-2100 COPENHAGEN Ø, DENMARK